Linear Models

Examples of Tasks from Course 1, Unit 3

What Solutions are Available?

Lesson 1: page 174, Modeling Task 6; page 176, Organizing Task 3; page 178, Reflecting Task 1

Lesson 2: page 189, Modeling Task 3; page 190, Organizing Task 1; page 191, Organizing Task 5; page 193, Extending Task 4; page 203, Modeling Task 3; page 205, Organizing Task 3; page 206, Reflecting Task 4

Lesson 3: page 216, Modeling Task 2; page 218, Organizing Task 3; page 219, Extending Task 2; page 223, Modeling Task 3; page 224, Organizing Task 2; page 230, Organizing Task 1; page 239, Modeling Task 3; page 241, Extending Task 1

These tasks are selected with the intent of presenting key ideas and skills. Not every answer is complete, so that teachers can still assign these questions and expect students to finish the tasks. If you are working with your student on homework, please use these solutions with the intention of increasing your child’s understanding and independence.

As you read these selected homework tasks and solutions, you will notice that some very sophisticated communication skills are expected. Students develop these over time. This is the standard for which to strive. See Research on Communication.

The Algebra page or the Scope and Sequence might help you follow the conceptual development of the ideas you see in these examples.

Main Mathematical Goals for Unit 3

Upon completion of this unit, students should be able to:

- recognize situations in which key variables change at a constant rate
- express and interpret those patterns of change in data tables, slopes and intercepts of straight line graphs, and equations in the form $y = a + bx$
- use techniques for solving linear equations and inequalities that arise in science and business problems

Students make use of graphing calculators to accomplish tasks such as evaluating and solving, as well as doing these tasks by hand. Students will frequently use the language “rate of change” when talking about the “slope” of a linear function. Also note that the letter “b” is not reserved almost exclusively for the y-intercept as you may have experienced in your mathematics program. The $y = a + bx$ form emphasizes the modeling or statistical approach to linear functions where $a$ is the starting value or y-intercept and $b$ is the rate of change. One goal is to help students develop flexible use of variables. Thus, you will notice many different letters representing linear functions throughout this unit and the entire Core-Plus Mathematics curriculum.
Selected Homework Tasks and Expected Solutions

(These solutions are for problems in the book with 2003 copyright. If a student is using a book with an earlier copyright, you may notice that the problems don’t match exactly, although the intent of the problems should be the same.)

Lesson 1, page 174, Modeling Task 6

a. The graph shows a trend which is reasonably close to linear.

This unit is about linear models, but since linear equations are developed from data, the pattern in the graphs of the data is approximately linear.

b. During in-class work on Lesson 1, Investigation 3, students saw that it made sense to choose a line that contains the point \((\bar{x}, \bar{y})\). Thus, since \(\bar{x} = 218.5\) and \(\bar{y} = 138.1\), the line should pass through this point. When you substitute \((218.5, 138.1)\) into the equations offered, you find that neither equation contains this point, so neither line would go through the point \((\bar{x}, \bar{y})\).

c. The line \(y = 0.5x + 25\) seems to be a better fit, since the data points are fairly close to this line, and they are also spread about evenly on either side of the line. The line \(y = 0.7x + 8\) is not a good fit, because most of the points are below that line. If this line (or equation) were used to predict interceptions from touchdowns, it would consistently make predictions that are too high. The other line would sometimes be too high, sometimes low, and would generally be quite good for making predictions.

d. To calculate the mean absolute error, students have to find out by how much each data point misses the line and average these errors. A calculator can make this shorter, but with only 10 data points it is also possible to do this by hand. Doing this calculation (it is not completed here) confirms that the mean absolute error is smaller for \(y = 0.5x + 25\), which visually looked like a better fit also.
The lists in a calculator could be set up as shown below in order to calculate the mean absolute error for these two lines.

<table>
<thead>
<tr>
<th>List</th>
<th>Description</th>
<th>Values</th>
<th>Calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1</td>
<td>x (touchdown data)</td>
<td>232</td>
<td></td>
</tr>
<tr>
<td>L2</td>
<td>y (interception data)</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>L3</td>
<td>0.5(L1) + 25 (predicted y using (y = 0.5x + 25))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.5(232) + 25 = 141</td>
<td>etc.</td>
</tr>
<tr>
<td>L4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L5</td>
<td>0.7(L1) + 8 (predicted y using (y = 0.7x + 8))</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.7(232) + 8 = 170.4</td>
<td>etc.</td>
</tr>
<tr>
<td>L6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L7</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ L_7 = |L_2 - L_3| \text{ (absolute error from } y = 0.5x + 25) \]
\[ L_8 = |L_2 - L_5| \text{ (absolute error from } y = 0.7x + 8) \]

\[ e. \quad y = 0.5(120) + 25 = 85 \text{ interceptions.} \]

Lesson 1, page 176, Organizing Task 3

a. \( \frac{1}{4} , \frac{3}{12} , \frac{5}{20} , \frac{7}{28} , \frac{9}{36} \). All the ratios are equivalent to 1:4.

b. \( \frac{6}{x} = \frac{1}{4} \)

c. \( \frac{6}{x} = \frac{6}{24} \) so \( x \) must be 24.

d. • 1 cm on the transparency is equivalent to 4 cm on the screen. 8 cm on the transparency is equivalent to \( x \) on the screen. \( \frac{1}{4} = \frac{8}{x} \), \( \frac{1}{4} = \frac{8}{32} \) so \( x = 32 \text{ cm} \).

• \( \frac{1}{4} = \frac{x}{16} \), etc.


Students should have had experience solving proportions in middle school. But even if they have not had this experience they can reason by using equivalent ratios.
Lesson 1, page 178, Reflecting Task 1

a. The high and low points were joined. This makes quite a good fit visually. However, as students gain in experience and sophistication, this particular fit will be seen to have flaws. Most points below the line are on the left. Most points above the line are on the right. This kind of pattern indicates a better fit is possible.

b. Farthest left and farthest right points were joined. Visually this is a poor fit. Most of the data is below the line.

c. This appears to hit the maximum number of points. This is not a good fit, because some of the points are very far away from the line, so that these points will add considerably to the mean absolute error.

d. This line fits the data well. The points are randomly distributed around the line, no point is very far away from the line, and there are about the same number of points above the line as below.

e. This line has the same number of points on either side, but the line does not follow the trend of the data at all.

Lesson 2, page 189, Modeling Task 3

a.

The point of this task is to have students confront some of the strategies that they may have initially thought as valid for fitting a line to data.
b. The above graph has the line of best fit as found by a calculator. Students will not use this strategy yet, but their lines should look something like the line shown. When they try to find an equation for what they have drawn they will probably try to estimate the slope and intercept, and perhaps enter the data in lists and their equations in the Y= menu on their calculators to see if it resembles what they have drawn. Students should have an equation like \( M = 230 - 15W \).

c. Using the above equation, the predictions are \( M = 230 - 15(10) = $80 \) after 10 weeks and \( M = 230 - 15(15) = $5 \) after 15 weeks. After 20 weeks, \( M = 230 - 15(20) = -70 \), or the account is $70 “in the hole.”

d. The table values decrease quite consistently by about $15 for 1 week. There is no starting value in the table for week 0, but we can guess this must be about $225 or $230. The graph of the line “starts” at (0, 230) and falls by 15 per week. These clues appear in the equation; the start appears as the constant, and the rate of decrease appears as the coefficient of the \( W \). This all says that Jose spends about $15 a week, starting with a balance of $230.

**Lesson 2, page 190, Organizing Task 1**

Students will have to pay attention to slopes to be successful here. Since there is no scale on the \( x \)-axis the \( y \)-intercepts are not identifiable. Trying to identify these lines by using only the first point in the table is not going to be an effective strategy.

a. Table a shows a constant rate of increase of 1. This table could represent graph A, B, or C. Since Table a will contain the origin, (0, 0), Table a must represent graph C.

b. Table b shows a constant decrease of 0.5, starting at (0, 10). This must represent graph E.

c. Table c shows a constant rate of increase of 1, the same as Table a. Thus, the graph for Table c must be parallel to the graph for Table a. Table c must represent graph B because the “start” for Table c is higher than for Table a.

d. Table d does not have a constant rate of increase, so it must represent a curve not a line, giving graph D.

*In the next investigation, students meet a more formal way of finding an equation for an existing line.*
e. Table e has a constant rate of increase of 2, which is higher than the rate of increase for Tables a and c. Thus, the graph must be steeper and so this table represents graph A.

Lesson 2, page 191, Organizing Task 5

a.  

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4 + 2(0) = 4</td>
</tr>
<tr>
<td>1</td>
<td>4 + 2(1) = 6</td>
</tr>
<tr>
<td>2</td>
<td>etc.</td>
</tr>
</tbody>
</table>

- The table shows that the y values increase by 2 as x increases by 1. If the “first” y value is 4. The NEXT y value is 6.
- NEXT = NOW + 2, starting at 4
- There’s a constant rate of increase of 2 to get to the next y value. (This assumes that the table has been set up with x values increasing by 1.)

b. The y values would drop by 6 as x increases by 1. The graph would slope down with a slope of –6 (or $\frac{-6}{1}$ if students want to use the $\frac{\Delta y}{\Delta x}$ form).

Lesson 2, page 193, Extending Task 4

a. 

It is important that students grasp this NOW-NEXT form, not only because it foreshadows work to be done later with sequences, but also because this form most naturally connects to the table.
b. Students will either make an estimate of the rate of increase from the table or the graph. There is no “start” given, but it makes sense to assume that when volume is zero, mass is also zero. So the equations are $\text{NEXT} = \text{NOW} + 0.9$, starting at 0, for oil, and $\text{NEXT} = \text{NOW} + 3.5$, starting at 0, for diamonds.

c. Students should be able to use the clues about rate and intercept from the $\text{NOW}-\text{NEXT}$ equations to form the equations.

- $M_{\text{oil}} = 0.9V$
- $M_{\text{diamonds}} = 3.5V$

d. Density and slope and coefficient of $V$ are the same. It is not generally true that if you divide a $y$ value by the corresponding $x$ value you get the slope. It only works in cases where the $y$-intercept is zero.

e. The table values increase at different rates and the coefficients of $V$ are different in the two equations.

f. • Mary Jo’s gemstone has a mass of 1.1 grams.
• The volume of the stone could be determined by displacement. Drop the stone into a graduated cylinder of water, measuring the height of the water, before and after the stone is added. The difference of these two measurements is the volume of the stone.
• These values do not fit the model for diamonds, so the stone is not a diamond.

Lesson 2, page 203, Modeling Task 3

a. The point of producing models, whether these are graphical or symbolic, is to be able to use the pattern to predict other values.
b. The points to join are given this time so that students will all be doing the same calculation. Generally, students will select two points they are sure are on the line for this procedure. The
slopes is \( \frac{160 - 90}{18 - 10} = \frac{70}{8} \approx 8.8 \). This means the equation is \( C = a + 8.8F \). To find the intercept, we must substitute a pair of values into this equation that we know must be on the line. We could choose either (18, 160) or (10, 90). Using the latter, \( 90 = a + 8.8(10) \), so \( a = 2 \). The \( b \) value tells you that there is an increase of about 8.8 calories for every 1 gram of fat added. The \( a \) value tells you that if there are 0 grams of fat, then there are 2 calories from fat. This does not make sense. The error comes in because we have observational data to work with, which may not be completely accurate. Also, we chose two points to anchor the line. The choice of these points may not have given the very best fit. Since it is close to zero, there is probably no need for concern.

c. Approximately 116 calories

d. The linear regression model is the best fit line. The graphing calculator is programmed with an algorithm that calculates values of \( a \) and \( b \) that will minimize the square errors. This means that by acknowledging that all the data will not fit exactly to a line, we then choose a line so that when you add up all the square distances between the line and the actual data points, this total is a minimum. On most calculators, the steps are to enter the data in two lists, say \( L_1 \) and \( L_2 \), then choose from the Statistics menu \textbf{linreg}(L_1,L_2). Students should have a technology tip in their toolkit on the steps to follow. In this case, the equation is
\( C = 1.83 + 8.92F \).

Lesson 2, page 205, Organizing Task 3
a. Students need to identify that the slope is positive, and the \( y \)-intercept is negative. The line must pass through quadrants III, IV, and I. (See the graph at the right.)

b–d. Students should be able to sketch these by using the slope and location of the \( y \)-intercept as was done in Part a.
Lesson 2, page 206, Reflecting Task 4

a. This is a quick method but not appropriate if you can’t see the y-axis to make a good estimate of the y-intercept, or if the scale on the axes makes it difficult to estimate the slope.

b. This would give a good estimate of the slope, but it could be hard to estimate the y-intercept.

c. This is slower but gives accurate results, if the points chosen are really on the line.

d. This is easy and accurate. It may be too time consuming if the data set is large. If a graph is already available, a quick estimate of the best fit line may be appropriate. If all that is available is the raw data in a table, then the calculator algorithm for the line of best fit is the best method.

Lesson 3, page 216, Modeling Task 2

a. **Soft Drink Data**

![Graph of Soft Drink Data](image)

b. The slope is about –10; this means that as drinks are sold, the amount of soft drink left in the supply tank decreases at a rate of 10 ounces per drink. The size of each drink is approximately 10 ounces.
c. (0, 2,500) This point tells us that the tank started out with 2,500 ounces.

d. Students should use their answers from Parts b and c to write this equation.

e. • $1,200 = 2,500 - 10N$, so $N = 130$. This means that about 130 drinks have been sold when the level is at 1,200 ounces. On the graph, find (?, 1,200).
    • $L = 2,500 - 10(125)$, so $L = 1,250$. etc.
    • $2,500 - 10N \geq 1,705$, so $N \leq 75$. etc.

Lesson 3, page 218, Organizing Task 3

a. Graph the rule. \textcolor{red}{\text{TRACE}} to find where $y = c$ on the line. The solution is the corresponding value of $x$. Another method uses two lines, the line $y = a + bx$ and the line $y = c$. In this case, use \textcolor{red}{\text{intersect}} to find the point (?, c).

b. As above but this time you want to find $x$ values when the $y$ values are greater than or equal to $c$. This time there are many solutions for $x$. For each of these $x$ values, the corresponding $y$ value is greater than or equal to $c$. Two examples are shown.

The point of the question is to have students make sense of the solving procedure. If they always have the equation given to them, they will concentrate on routine steps and not on what the equation means and what the solution means. So, here they have to write the equation that shows a relationship between $L$ and $N$, and then use the graph of the equation to find values of $L$ and $N$. This keeps the focus on relationships, a functions approach that will be extremely useful as more complex models are added.
c. As above but this time you want to find the intersection of two lines, \( y = a + bx \) and \( y = c + dx \). There is only one \( x \) value for the solution. Students should draw a diagram illustrating this solution. Students should also consider cases where the lines do not intersect (no solution) and where the lines are the same (all values of \( x \) will satisfy the equation).

d. As above but you want the \( x \) values that go with the part of the line \( y = a + bx \) which is below the line \( y = c + dx \), that is the \( y \) values for \( a + bx \) are less than the \( y \) values for \( c + dx \). Students should consider different relationships between the two lines and identify the appropriate solution set in each case.

Lesson 3, page 219, Extending Task 2

a. We want the \( x \) values that correspond to the parts of the line \( y = x + 3 \) which are above, or intersecting with, the curve \( y = x^2 - 3 \). That is, the \( y \) values on the line are greater than, or equal to, the \( y \) values on the curve. \(-2 \leq x \leq 3\).

b. As above, but now we want the \( x \) values that correspond to the parts of the line \( y = x + 3 \) which are below the curve. \( x < -2 \) or \( x > 3 \).

Lesson 3, page 223, Modeling Task 3

a. In this plan, Jeff and Mary get $5 a game plus $0.40 for each program sold.

b. Students are expected to solve this by hand. The solution is \( S = 33.3 \). The solution gives the number of programs for which the two plans would produce the same pay.

c. Students are expected to use the graph to solve \( 10 + 0.25S \geq 5 + 0.4S \). This tells you when \( P_1 \geq P_2 \), that is for how many programs the first plan would be better, or when the two plans would give the same pay.

d. Students need to know the values of \( S \) for which \( 10 + 0.25S = 0.75S \) and \( 5 + 0.4S = 0.75S \). Notice that students can use this graphing method to solve any equation or inequality.

Students could solve these equations by hand or by graphing. They should be making judgments about the efficiency of each strategy.
Lesson 3, page 224, Organizing Task 2

a. First subtract $a$ from both sides, then divide both sides by $b$.

b. First subtract $a$ from both sides, then divide both sides by $-b$.

c. First add $a$ from both sides, then divide both sides by $b$.

Lesson 3, page 230, Organizing Task 1

a. 

b. Women: $y = -0.27911x + 79.10660$
   Men: $y = -0.16918x + 64.30926$
   Note: The data was entered so 1900 corresponds to 0, 1912 corresponds to 12, etc.

c. Women have shown greater improvement over the years displayed in the table. One can observe this fact from the table: the women’s time has decreased approximately 28.37 seconds, or 35% since 1912, while the men’s has decreased 15.1 seconds, or 24% in that same time period.
   The linear regression lines show the same result. The line representing women has a more negative slope, showing that the winning time is decreasing more quickly.

d. The regression lines intersect at (134.6, 41.5). Students could also solve by hand:
   $-0.27911x + 79.10660 = -0.16918x + 64.30926$. 

This question gives students an opportunity to generalize the steps they have been using to solve linear equations.

This question not only connects the data unit to this algebra unit, but asks students to make sense of the solution.
e. The ordered pair representing the intersection of the two lines suggests that if the Olympic 100-meter swim takes place in 2036, the winning time for both men and women will be approximately 41.5 seconds. The lines are not a good predictor for anything but the near future. For example, according to the line modeling women’s times, in the year 2196 it will take –0.16 seconds to swim 100 meters.

Lesson 3, page 239, Modeling Task 3

a. \[ C = 5,000,000 + 4.75N; \quad I = 53.50N \]

b. \[ P = Income - Cost = 53.50N - (5,000,000 + 4.75N). \] Other equivalent equations are \[ P = 53.50N - 5,000,000 - 4.75N \] and \[ P = 48.75N - 5,000,000. \]

c. Students could use graphs and tables to show that all three variations give the same values for \( P \). They should also be able to explain why parentheses are necessary in the first form, and that removing the parentheses means both terms must now be subtracted, and that combining the like terms in the third version makes sense, since it means that the profit is $48.75 per disc.

Lesson 3, page 241, Extending Task 1

a. Students can use their calculator tables to check their intuitions about these expressions. They are not equivalent.

b. Not equivalent

Students are not using properties formally at this time. They will put formal names on these properties in Organizing Task 1 on page 239 and more fully in Course 3.

Although students have not yet been told the formal names of properties that they might use on these expressions, they are developing some symbol sense about parentheses and subtractions.