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UNIT 6

Surfaces and Cross Sections

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CAT scans and MRI (magnetic resonance imaging) are remarkable technologies that provide two-dimensional renderings of a brain or other organs of the human body. How to represent three-dimensional objects in two dimensions on a computer monitor or on paper is a fundamental question in geometry and its applications.

In this unit, the focus will be primarily on reasoning with representations of surfaces of three-dimensional objects such as those of a distant planet and of cross sections of three-dimensional objects such as those that are so useful in medical imaging.

In the two lessons of the unit, you will extend your skills in visualizing surfaces and cross sections and representing those objects, both geometrically and algebraically, through the use of three-dimensional coordinate methods.

Lessons

1. **Representing Three-Dimensional Objects**
   Represent three-dimensional surfaces with contour diagrams and identify and sketch cross sections of three-dimensional objects, including conic sections. Identify and sketch graphs of conic sections represented algebraically and write equations matching graphs of conic sections.

2. **Equations for Surfaces**
   Extend ideas of coordinate representation and methods in two dimensions to three dimensions. Visualize, describe, and sketch surfaces represented by equations and identify and sketch surfaces of revolution and cylindrical surfaces.
**Summarize the Mathematics**

The equation \( Ax + By + Cz = D \), where not all \( A, B, \) and \( C \) are zero, is a first-degree (or linear) equation in \( x, y, \) and \( z \).

- **a** What is the graph of an equation of this form?
- **b** How can you quickly sketch the graph?
- **c** What is the nature of each cross section of the graph of \( Ax + By + Cz = D \)?
- **d** How are the equations of planes parallel to coordinate planes special cases of \( Ax + By + Cz = D \)?

*Be prepared to explain your conclusions to the entire class.*

**Check Your Understanding**

Sketch the graph of each equation on a three-dimensional coordinate system.

- **a** \( 2x - y + 3z = 6 \)
- **b** \( x + y - 2z = 30 \)
- **c** \( x - 4y = 8 \)

**Investigation 3** — **Surfaces Defined by Nonlinear Equations**

In Investigation 2, you found that your understanding of linear equations made sketching planes defined by linear equations of the form \( Ax + By + Cz = D \) much easier. Similarly, to sketch surfaces defined by nonlinear equations, you can draw on your understanding of curves, particularly the conics, in a coordinate plane. As you work on the following problems, look for answers to this question:

*What strategies are helpful in sketching surfaces and judging the reasonableness of technology-produced graphs defined by nonlinear equations in three variables?*

*How can you describe symmetries of three-dimensional figures?*
Re-examine the ellipse and corresponding surface, called an *ellipsoid*, from the beginning of this lesson.

![Ellipsoid Diagram](image1.png)

**a.** Is the ellipse symmetric with respect to the x-axis? With respect to the y-axis? Explain your reasoning.

**b.** If a point \(A(a, b)\) is on the ellipse, name the coordinates of at least two other points on the ellipse.

**c.** The ellipsoid above is *symmetric with respect to the xz-plane*. If a point \(A(a, b, c)\) is on the ellipsoid, then the point \(A'(a, -b, c)\) is also on the ellipsoid.

   **i.** Is the ellipsoid *symmetric with respect to the xy-plane*? If so, what are the coordinates of the point that is symmetric to point \(A\) with respect to the xy-plane?

   **ii.** Is the ellipsoid *symmetric with respect to the yz-plane*? If so, what are the coordinates of the point that is symmetric to point \(A\) with respect to the yz-plane?

**Now consider the algebraic representation for the ellipse above. Scales on the axes are 1.**

**a.** Explain why an equation for the ellipse is \(\frac{x^2}{9} + \frac{y^2}{16} = 1\).

**b.** Note that when you substitute \(-x\) for \(x\) in the equation in Part a, you get \(\frac{(-x)^2}{9} + \frac{y^2}{16} = 1\) or \(\frac{x^2}{9} + \frac{y^2}{16} = 1\), which is the same as the original equation.

   **i.** Why can you use this fact to conclude that the graph of the equation is symmetric with respect to the y-axis?

   **ii.** How could you determine that the graph of \(\frac{x^2}{9} + \frac{y^2}{16} = 1\) is symmetric with respect to the x-axis by only reasoning with the symbolic form?

**In the Think About This Situation at the beginning of this lesson, you may have conjectured that an equation for the ellipsoid is \(\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1\).** Explain why this graph-equation match makes sense in terms of each of the following.

**a.** \(x-, y-,\) and \(z\)-intercepts
b. Cross sections determined by the coordinate planes and planes parallel to the coordinate planes

i. What are the equation and shape of the cross section of the graph of \( \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1 \) determined by the yz-plane? The xz-plane? The xy-plane?

ii. What are the equation and shape of the cross section determined by setting \( z = 2 \)? By setting \( y = 2 \)? By setting \( x = 1 \)?

c. Symmetry

i. What symmetry of the graph of \( \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1 \) is implied by the fact that
\[
\frac{x^2}{9} + \frac{y^2}{16} + \frac{(-z)^2}{4} = \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1?
\]

ii. How could you test for 180° rotational symmetry of the graph of \( \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1 \) about the x-axis by reasoning with the equation itself?

iii. How would you test for symmetry with respect to the yz-plane by reasoning with the equation itself?

4 Now consider in more detail 180° rotational symmetry in two and three dimensions.

a. Does the ellipse on page 465 appear to have 180° rotational (half-turn) symmetry about the origin? Explain.

b. How could you determine that the graph of \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \) has half-turn symmetry about the origin by only reasoning with the symbolic form?

c. The ellipsoid in Problem 3 has half-turn symmetry about the z-axis. If a point \( A(a, b, c) \) is on the ellipsoid, then the point \( A'(−a, −b, c) \) is also on the ellipsoid.

i. Does this ellipsoid have half-turn symmetry about the x-axis? If so, what are the coordinates of the point that is symmetric to point \( A \) under this half-turn?

ii. Does this ellipsoid have half-turn symmetry about the y-axis? Explain your reasoning.

iii. How could you test for half-turn symmetry of the graph of \( \frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1 \) about the x-axis by reasoning with the equation itself?

d. The paraboloid at the right is defined by the equation \( x^2 − y + 4z^2 = 0 \).

i. Does the paraboloid have half-turn symmetry about the y-axis? How could you test for half-turn symmetry about the y-axis by reasoning with the equation itself?

ii. Does the paraboloid have half-turn symmetry about the x-axis? How could you test for half-turn symmetry about the x-axis by reasoning with the equation itself?
Shown below is a technology-produced graph of $x^2 + y^2 - z = 0$. Horizontal cross sections (and perhaps a contour diagram) can be used to explain why the graph is correct. That is, it is a bowl shape rather than a cone.

a. The horizontal cross sections are determined by planes with equations of the form $z = c$. Find the equations for the horizontal cross sections at intervals of $c = 0, 1, 2, 3, \text{ and } 4$.

b. Describe the sequence of graphs of the equations you found in Part a.

c. What does this tell you about the shape of the graph and its correctness?

d. Explain why vertical cross sections of the graph of $x^2 + y^2 - z = 0$ are determined by equations $y = c$ or $x = c$.

e. Find the equation of several vertical cross sections at 1-unit intervals. Explain how those equations are revealed in the shape of the graph above.

Use horizontal cross sections at 1-unit intervals (and perhaps a contour diagram) to help you explain why the graph of $x^2 + y^2 - z^2 = 0, z \geq 0$, is a cone. Sketch a graph of the cone.

Use analysis of intercepts, cross sections, and symmetry to help you match each equation with one of the following surfaces.

LESSON 2 • EQUATIONS FOR SURFACES 467
Use analysis of intercepts, cross sections, and symmetry to help you visualize and sketch surfaces given by each of the following equations. Describe each surface. Compare your surfaces and descriptions to those of your classmates and resolve any differences.

a. \( x^2 + y^2 - z^2 = 0 \)  

b. \( x^2 + 4y^2 - z^2 = 4 \)  
c. \( x^2 + 2y^2 + 3z = 6 \)  
d. \( x^2 + y^2 - z = 0 \)

Graphing surfaces in three dimensions can be time consuming and difficult. Three-dimensional graphs are more easily produced using computer software with three-dimensional graphing capability. Another advantage of these technologies is that they may permit you to view the surface from different angles. Software with implicit graphing capabilities can graph equations such as \( 4x + 5y = 7 \) in two dimensions and equations such as \( x^2 - y + z^2 = 1 \) (Problem 8) in three dimensions.

Computer-produced graphs of surfaces such as that on the preceding page and below are produced by drawing cross sections parallel to two coordinate planes, creating a mesh. The fineness of the mesh is determined by how close adjacent cross sections are to each other. In a “smooth” rendering of a surface by computer graphic software as shown below, color and shading are used to fill in the quadrilaterals of the mesh and then a smoothing algorithm is applied so the surface appears realistic.
Three-dimensional graphing software typically graph equations of the form \( z = f(x, y) \). This notation means that \( z \) is a function of two variables, \( x \) and \( y \).

**a.** The technology-produced display below shows the graph of \( z = f(x, y) = \sqrt{1 - x^2 + y} \). Explain differences between this graph and the one you sketched in Problem 8 Part a.

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**b.** Being able to judge the reasonableness of technology-produced graphs is as important when working in a three-dimensional coordinate system as when working in a two-dimensional coordinate system. Explain why it is reasonable that the plot shown in the screen below is that of \( x^2 - y^2 - z = 0 \).

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**c.** Use three-dimensional graphing software to graph each of the following equations. Describe the general shape of each surface. Experiment with viewing the surface from different angles.

i. \( z = \frac{x^2y - y^2x}{400} \)

ii. \( z = 2x - 3y + 3 \)

iii. \( z = \sqrt{25 - x^2 - y^2} \)

iv. \( z = \frac{x^2y^2 - y^3x}{400} \)
Summarize the Mathematics

In this investigation, you examined effective strategies for graphing nonlinear equations involving three variables. Suppose you are given the equation of a surface in variables $x$, $y$, and $z$.

a. How would you find the $x$-, $y$-, and $z$-intercepts of the surface?

b. How would you find the traces? Cross sections? Explain how traces and cross sections are similar and different.

c. Explain how you could use the equation to determine if the surface has any symmetries.

d. How would you check the reasonableness of a technology-produced three-dimensional graph?

Be prepared to explain your procedures to the entire class.

Check Your Understanding

Sketch and describe each of the following surfaces.

a. $(x - 2)^2 + y^2 + (z - 1)^2 = 25$

b. $x^2 + y^2 + z = 4$

Investigation 4 Surfaces of Revolution and Cylindrical Surfaces

Graphs of equations in three variables are surfaces. Some of these surfaces can be generated by rotating (or revolving) a curve about a line, sweeping out a surface of revolution. The line about which the curve is rotated is called the axis of rotation. A table leg or lamp base turned on a lathe has a surface of revolution. A potter using a potting wheel makes surfaces of revolution.

Some common surfaces can be thought of as surfaces of revolution. As you work on the problems of this investigation, look for answers to this question:

What strategies can be used to identify and sketch surfaces of revolution and cylindrical surfaces?