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In previous units, you have used equations, tables, and graphs to investigate linear, exponential, polynomial, and periodic patterns of change. You have used coordinates and matrices to model geometric change in position, size, and shape. In many situations, it is also important to understand step-by-step sequential change, such as yearly change in population or hourly change in antibiotic concentration after taking medication.

Recursion and iteration are powerful tools for studying sequential change. You have already used recursion and iteration when you used NOW-NEXT rules to solve problems. In this unit, you will study sequential change more fully. The concepts and skills needed are developed in the following three lessons.

### Lessons

1. **Modeling Sequential Change Using Recursion and Iteration**
   - Represent and solve problems related to sequential change, using subscript and function notation and technological tools, such as spreadsheets.

2. **A Recursive View of Functions**
   - Analyze linear, exponential, and polynomial functions from a recursive point of view, specifically through the study of arithmetic and geometric sequences and finite differences tables.

3. **Iterating Functions**
   - Investigate the general process of iterating functions, and completely analyze the behavior of linear functions when they are iterated, including an analysis of slope to identify attracting and repelling fixed points.
Modeling Sequential Change Using Recursion and Iteration

Wildlife management has become an increasingly important issue as modern civilization puts greater demands on wildlife habitat. Tracking annual changes in the size of wildlife population is essential to effective management. In previous units, you have used NOW-NEXT rules or formulas to model situations involving sequential change. Now you will examine sequential change more fully.
In this unit, you will use recursion and iteration to represent and solve problems related to sequential change, such as year-to-year change in a population or month-to-month change in the amount of money owed on a loan. You will use subscript notation and function notation. Spreadsheet software will be used to help with the analysis.

Investigation 1: Modeling Population Change

The first step in analyzing sequential change situations, like the fish population situation, is to build a mathematical model. As you work on the problems of this investigation, look for answers to this question:

How can you construct and use a mathematical model to help you analyze a changing fish population?

As you have seen before, a typical first step in mathematical modeling is simplifying the problem and deciding on some reasonable assumptions. Three factors that you may have listed in the Think About This Situation discussion are initial fish population in the pond, annual growth rate of the population, and annual restocking amount, that is, the number of fish added to the pond each year. For the rest of this investigation, use just the following assumptions.

• There are 3,000 fish currently in the pond.
• 1,000 fish are added at the end of each year.
• The population decreases by 20% each year (taking into account the combined effect of all causes, including births, natural deaths, and fish being caught).
Using these assumptions, build a mathematical model to analyze the population growth in the pond as follows.

a. Estimate of the population after one year. Estimate the population after two years. Describe how you computed these estimates. What additional details did you assume to get the answers that you have?

b. Assume that the population decreases by 20% before the 1,000 new fish are added. Also assume that the population after each year is the population after the 1,000 new fish are added. Are these the assumptions you used to compute your answers in Part a? If not, go back and recompute using these assumptions. Explain why these assumptions are reasonable. These are the assumptions you will use for the rest of this analysis.

c. Write a formula using the words NOW and NEXT to model this situation as specified in Part b.

d. Use the formula from Part c and the last-answer feature of your calculator or computer software to find the population after seven years. Explain how the keystrokes or software features you used correspond to the words NOW and NEXT in the formula.

Now think about the patterns of change in the long-term population of fish in the pond.

a. Do you think the population will grow without bound? Level off? Die out? Make a quick guess about the long-term population. Compare your guess to those made by other students.

b. Determine the long-term population by continuing the work that you started in Part d of Problem 1.

c. Explain why the long-term population you have determined is reasonable. Give a general explanation in terms of the fishing pond ecology. Also, based on the assumptions above, explain mathematically why the long-term population is reasonable.

d. Does the fish population change faster around year 5 or around year 25? How can you tell?

What do you think will happen to the long-term population of fish if the initial population is different but all other conditions remain the same? Make an educated guess. Then check your guess by finding the long-term population for a variety of initial populations. Describe the pattern of change in long-term population as the initial population varies.

Investigate what happens to the long-term population if the annual restocking amount changes but all other conditions are the same as in the original assumptions. Describe as completely as you can the relationship between long-term population and restocking amount.
Describe what happens to the long-term population if the annual decrease rate changes but all other conditions are the same as in the original assumptions. Describe the relationship between long-term population and the annual decrease rate.

Now consider a situation in which the fish population shows an annual rate of increase.

a. What do you think will happen to the long-term population if the population increases at a constant annual rate? Make a conjecture and then test it by trying at least two different annual increase rates.

b. Write formulas using NOW and NEXT that represent your two test cases.

c. Do you think it is reasonable to model the population of fish in a pond with an annual rate of increase? Why or why not?

**Summarize the Mathematics**

In this investigation, you constructed a NOW-NEXT formula to model and help analyze a changing fish population. Consider a population change situation modeled by \( \text{NEXT} = 0.6 \times \text{NOW} + 1,500 \).

a. Describe a situation involving a population of fish that could be modeled by this NOW-NEXT formula.

b. What additional information is needed to be able to use this NOW-NEXT formula to predict the population in 3 years?

c. What additional information is needed to be able to predict the long-term population?

Consider the following variations in a fish-population situation modeled by a NOW-NEXT formula like the one above.

i. If the initial population doubles, what will happen to the long-term population?

ii. If the annual restocking amount doubles, what will happen to the long-term population?

iii. If the annual population decrease rate doubles, what will happen to the long-term population?

How would you modify \( \text{NEXT} = 0.6 \times \text{NOW} + 1,500 \) so that it represents a situation in which the fish population increases annually at a rate of 15%? What effect does such an increase rate have on the long-term population?

Be prepared to explain your ideas to the class.