Lesson 2

Polynomials and Factoring

A polynomial function is a power function or the sum of two or more power functions, each of which has a nonnegative integer power. Because polynomial functions are built from power functions, their domains are all real numbers. The graph of a polynomial function is smooth and unbroken. This important property of polynomial functions makes them particularly useful for transforming one image into another during special effects in movies, for creating fonts for computers, and for approximating solutions to otherwise uncomputable problems.

In this lesson, you will explore some of the advantages of expressing polynomials in different equivalent forms.

Think About This Situation

Polynomials can be written symbolically in several different forms. Consider the three forms of one polynomial shown below.

Standard Form: \( P(x) = 5x^4 + 10x^3 - 65x^2 - 70x + 120 \)

Factored Form: \( P(x) = 5(x - 1)(x + 2)(x - 3)(x + 4) \)

Nested Multiplication Form: \( P(x) = ((5x + 10)(x - 65)x - 70)x + 120 \)

\( \text{a} \) What mathematical questions could be most easily answered using:
- the standard form?
- the factored form?
- the nested form?

\( \text{b} \) Can every polynomial be written in factored form? In nested multiplication form?
INVESTIGATION 1 Need for Speed

In the previous lesson, you may have noticed that the time needed to produce a curve on a calculator depends upon the complexity of the equation involved. This fact is very important to computer scientists, mathematicians, and engineers, for whom computation time is often a critical component of any project. Saving computer time saves money. Super computers such as the SGI Origin 2000 at the National Center for Super-computing Applications at the University of Illinois at Champaign–Urbana can cost between 2 and 3 million dollars a year to operate.

1. The form of a function rule can greatly influence the time it takes to calculate function values or produce its graph. In this activity, you will use your calculator to measure computation time for polynomial functions.

   a. Working in pairs, measure and record the time required to graph the function

   \[ y = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 \]

   with the viewing window set to \( 0 \leq x \leq 6 \) and \( -10 \leq y \leq 10 \). One person should watch the graph and the other should watch the time. Be sure to turn off any other functions or plots before doing this experiment.

   b. Next measure and record the time it takes to graph the function

   \[ y = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5) \]

   Be sure to turn off any other functions.

   c. Verify that the function rules in Parts a and b are equivalent.

   d. Which form produces the graph more quickly? What percentage of time is saved using this form?

   e. Computation time is measured in units called cycles. Suppose computing any power, such as \((1.2)^4\), uses twice as much computation time as does any multiplication, subtraction, or addition, which each require 1 cycle.

   - How many cycles are required when

   \[ y = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 \]

   is evaluated for a particular value of \( x \)?

   - How many cycles are required when the equivalent rule

   \[ y = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5) \]

   is evaluated?

   - How does this computation time for the factored form compare to the computation time for the standard polynomial form?
2. The factored form of a polynomial can reduce the computation time, but not all polynomials can be written as a product of linear factors. Another way a polynomial can be rewritten to avoid exponents is to use nested multiplication.

a. Verify that the polynomial \(5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1\) can be rewritten in nested form as (((((5x + 4)x + 3)x + 2)x + 1)x + 1).

b. Determine the percentage of computation time (in seconds) saved in graphing the function \(y = 5x^5 + 4x^4 + 3x^3 + 2x^2 + x + 1\) expressed in nested multiplication form rather than in standard polynomial form. Use the window \(-2 \leq x \leq 2\) and \(-40 \leq y \leq 40\).

c. How many cycles are required to compute each form of the polynomial? How is the cycle count related to your answer in Part b?

d. Investigate other viewing windows for the graph of the polynomial function. Explain why this polynomial cannot be written as a product of linear factors. Into how many linear factors can this polynomial be factored?

e. Rewrite the polynomial of Activity 1 using nested multiplication. How does the computation time for nested multiplication compare to the computation time for the factored form?

Checkpoint

The form of a polynomial influences computation speed as well as the information that can be determined by examining the form.

a. Write the polynomial function \(y = (x + 2)(x - 5)(x^2 + 1)\) in standard polynomial form; in nested multiplication form.

b. What information about the graph of a polynomial function can you get from the factored form? From the expanded or standard form? From the nested multiplication form?

Be prepared to discuss the importance of each form of a polynomial with your classmates.

Evaluating polynomials using nested multiplication is frequently called Horner’s Method after the English mathematician William George Horner (1786–1837). However, evidence was published in 1911 that Paolo Ruffini (1765–1822) had used the method 15 years before Horner. More recently, the method has been found over 500 years before either Ruffini or Horner in the works of Chinese mathematicians during the late Sung Dynasty: Li Chih (1192–1279), Chu Shih-chieh (1270–1330), Ch’in Chiu-shao (c. 1202–1261) and Yang Hui (c. 1261–1275).
On Your Own

Because evaluation of polynomial functions only requires addition, subtraction, and multiplication, they are often used to estimate nonpolynomial functions. Graph the function \( y = \sin x \) in the window \(-4 \leq x \leq 4\) and \(-2 \leq y \leq 2\). Then graph each of the following polynomial functions in the same viewing window.

i. \( y = x \)

ii. \( y = x - \frac{x^3}{3!} \)

iii. \( y = x - \frac{x^3}{3!} + \frac{x^5}{5!} \)

iv. \( y = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \)

a. Compare the graphs of the polynomial functions to the graph of the sine function.

b. Determine the zeroes of each polynomial function. How do these values compare to the zeroes of the sine function?

c. Write the next polynomial function approximation in this sequence.

d. Compare the time it takes your calculator to graph the fourth polynomial function above written in standard form and in nested multiplication form.

INVESTIGATION 2 Strategic Factors

Throughout your work with quadratic and higher degree polynomial functions, the factored form of the polynomials has played an important role. Historically, writing polynomials in factored form was important for finding solutions to equations. Problems involving the solution of polynomial equations date back to the time of the ancient Egyptians, Babylonians, and Greeks. The first systematic presentation of the solution of polynomial equations is attributed by some historians to a group of Arab mathematicians starting with al-Kwarizmi (c. 800–847).


a. She reports that a systematic solution of the equation \( x^2 + 10x = 39 \) was described by al-Kwarizmi in the early 800s.
   - Describe all the methods you know for solving this equation.
   - Solve this equation by reasoning with the symbolic form itself.

b. Several centuries earlier around 250 A.D., Diophantus of Alexandria solved the equation \( 630x^2 + 73x = 6 \). Using a method of your choice, find the solutions to this equation.

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2. Until the sixteenth century, the solution of polynomial equations involved systematic guessing-and-testing for possible linear factors involving integers.

a. Expand the polynomial \( P(x) = (x - a)(x - b)(x - c) \), where \( a, b, \) and \( c \) are given constants. Write the polynomial in standard form where the coefficients of each power of \( x \) are expressions involving \( a, b, \) and \( c. \)

b. What do you observe about the constant term? How could the constant term be used to identify possible factors of a polynomial?

c. What do you observe about the coefficient of the \( x^2 \) term?

d. How might you generalize your observations in Parts b and c so that they apply to \( n \)th-degree polynomials? Compare your generalization with those of other groups. Resolve any differences.

e. Use your generalization to solve the following equations.
   i. \( x^3 - 10x^2 + 27x - 18 = 0 \)
   ii. \( x^3 - 28x = 48 \)
   iii. \( x^3 - 4x = 0 \)

3. Rewrite each polynomial function in factored form using information from its graph and your generalization in Activity 2.

a. \( p(x) = x^3 + 9x^2 + 11x - 21 \)

b. \( y = x^4 + 2x^3 - 13x^2 - 14x + 24 \)

c. \( A = w^4 - 3w^3 + 2w^2 \)

d. \( s = r^4 + 2r^3 - 11r^2 - 12r + 36 \)

While searching for factors of a given polynomial, you probably used a fundamental relationship between the factors and the zeroes of the polynomial function.

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**Factor Theorem**

The linear expression \( (x - b) \) is a factor of a polynomial function \( f(x) \) if and only if \( f(b) = 0. \)

You will be asked to complete a proof of this theorem in Organizing Task 1 (page 400).

4. The method of Activity 2 can be generalized to find factors of the form \( (ax - b) \).

a. Without expanding completely, mentally determine the leading coefficient and constant term of the polynomial \( f(x) = (-1x + 1)(3x - 2)(-5x - 4). \)

b. What are the zeroes of the polynomial function in Part a? How are the zeroes related to divisors of the leading coefficient and of the constant term of the polynomial?
c. Use your observations from Parts a and b to factor each of the following polynomials:

i. \(4x^3 + 9x^2 - 4x - 9\)

ii. \(3x^3 + 11x^2 + 12x + 4\)

iii. \(x^3 - 2x^2 - 4x + 8\)

Your work in Activities 2 and 4 suggests the following useful theorem about possible rational zeroes of a polynomial function with integer coefficients.

5. You now have several ways to determine if a polynomial of the form \((x - a)\) is a factor of a polynomial \(p(x)\). In this activity, you will examine a method for finding the polynomial \(q(x)\) such that \(p(x) = (x - a)q(x)\).

a. Describe two ways of verifying that \((x + 1)\) is a factor of \(2x^3 + x^2 - 4x - 3\).

b. To find the polynomial which when multiplied by \((x + 1)\) gives \(2x^3 + x^2 - 4x - 3\), you can use a procedure similar to long division. Polynomial division is illustrated below. Explain how the term in bold below is chosen at each step.

\[
\begin{align*}
2x^2 - x - 3 \\
\underline{\phantom{2x^3 + 2x^2 - 3x - 3}} x + 1 \\
2x^3 + x^2 \\
\underline{-2x^3 - 2x^2} x + 1 \\
-x^2 - x \\
\underline{\phantom{-2x^3 - 2x^2} -3x - 3} x + 1 \\
-3x - 3 \\
\underline{\phantom{-2x^3 - 2x^2} -3x - 3} \\
0 \\
\end{align*}
\]

Therefore, \(2x^3 + x^2 - 4x - 3 = (x + 1)(2x^2 - x - 3)\).

c. Verify that \(2x^3 + x^2 - 4x - 3 = (x + 1)(2x^2 - x - 3)\).

d. Show that \(2x^3 + x^2 - 4x - 3\) can be written as a product of three linear factors.

e. The polynomial \(x^3 - 5x - 12\) is equivalent to \(x^3 - 0x^2 - 5x - 12\).

- Use polynomial division to show that \(x^3 - 5x - 12 = (x - 3)(x^2 + 3x + 4)\).
- Can \(x^3 - 5x - 12\) be written as a product of three linear factors? Explain your reasoning.
6. Factor each polynomial into polynomials of the smallest possible degrees.
   a. \( f(x) = x^3 - 2x^2 - 4x + 8 \)
   b. \( g(x) = x^4 - 3x^3 + 2x^2 \)
   c. \( h(x) = 2x^3 - 5x^2 - 28x + 15 \)
   d. \( p(x) = x^3 + 3x^2 + 4x + 2 \)
   e. \( q(x) = 8x^4 + 52x^3 + 66x^2 + 31x + 5 \)

7. In Lesson 1, you discovered that the multiplicity of a zero of a polynomial function was related to the shape of its graph near the zero.

   a. Compare the graphs of the polynomial functions in Activity 6 with their factorizations. Is the expected relationship between the shape of the graph, multiple zeroes, and repeated factors confirmed?

   b. Find a polynomial function whose graph matches the graph shown below.

   c. Use the connection between graphs, multiple zeroes, and repeated factors to help factor each of these polynomials into polynomials of the smallest degrees possible.
      i. \( y = x^3 - 7x^2 - 5x + 75 \)
      ii. \( y = x^3 + 4x^2 + x^3 - 10x^2 - 4x + 8 \)
      iii. \( y = x^3 + x^2 - 2x^3 - 5x^2 - 5x - 2 \)
      iv. \( y = x^4 - 4x^3 + 13x^2 - 36x + 36 \)
Checkpoint

Writing and interpreting polynomials in factored form has a long history as a central idea in algebraic thinking.

a. How is the solution of a polynomial equation \( p(x) = 0 \) related to the factored form of the polynomial \( p(x) \)?

b. What strategies can you use to find the factored form of a polynomial?

c. How does a repeated linear factor reveal itself in the graph of a polynomial function?

d. Explain why a polynomial of degree 7 cannot have just four linear factors (counting the repeated factors).

Be prepared to explain your strategies for, and thinking about, factoring polynomials to the class.

The quadratic formula can be used to solve second-degree polynomial equations. More complicated formulas exist for solving third- and fourth-degree polynomial equations. In the early 1800s, Paolo Ruffini, Neils Abel, and Evariste Galois found ways to show that it is not possible to solve fifth-degree equations by a formula. Since no general formula exists for finding roots of polynomial equations, mathematicians have developed systematic methods for estimating roots. One of these methods is investigated in the MORE set.

On Your Own

As you complete these tasks, think about the relationships among zeroes, factors, and graphs of polynomial functions.

a. Factor each polynomial into polynomials of the smallest degrees possible.

- \( f(x) = x^3 + 4x^2 - 11x + 6 \)
- \( g(x) = x^3 - 3x^2 - 10x \)
- \( h(x) = x^4 + 5x^3 + 7x^2 + 5x + 6 \)
b. Sketch the graph of a fifth-degree polynomial function that has one factor repeated twice and a second factor repeated three times. Write a symbolic rule that matches your graph.

c. Write an equation for the graph shown. The y-intercept is –6.

![Graph of a fifth-degree polynomial function]

d. Andrea used the calculator graph shown at the right to help her solve the equation $x^4 - x^3 - 27x^2 + 81x - 54 = 0$. She determined that the roots of the equation were $x = 1$ and $x = 3$. Yolanda, who was working in a group with Andrea, looked at the graph and said there had to be another root and found that –6 was also a solution.

- Verify that 1, 3, and –6 are all roots.
- How did Yolanda know there had to be another root?